International Conference on
Contemporary Issues in Engineering,
Agriculture, Applied Sciences and Humanities
(EAH-2019)

Application of Orthonormal Bernoulli's Polynomials for Solving Fractional Order Lane-Emden Type Differential Equations

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Abstract—*In this paper, an efficient method based on orthonormal Bernoulli's polynomials expansion, together with operational matrices of Caputo fractional derivative is proposed in order to solve the fractional Lane-Emden type differential equation. The orthonormal Bernoulli's polynomials method are generated here by dilation and translation of the classical orthonormal Bernoulli's polynomials. These functions and their associated properties are then applied to derive operational matrices of fractional derivative and integer order derivative. The operational matrices of fractional integrals are utilized to reduce the fractional Lane-Emden differential equation to a set of algebraic equations with unknown coefficients. Several examples are illuminated to reveal the validity and applicability of the proposed method.*

Keywords: *Orthonormal Bernoulli's polynomials; Fractional differential equation; Lane-Emden type equations; operational matrix of fractional integration; Numerical simulation.*

Mathematics Subject Classification: *34A08; 26A33; 34E05; 33F05; 26A33.*

1. Introduction

Fractional calculus is the outcome of multi-disciplinary endeavour that brought together mathematicians, physicists and engineers. This relationship created a flow of ideas that goes well beyond the construction of new transforms. The origin of fractional calculus can be traced back to the end of 17^{th} century, shortly after the development of classical calculus. The earliest systematic studies were attributed to Leibniz, Caputo, Hadamard, Fourier, Lioville and Riemann. Although fractional calculus is a natural generalization of calculus but it has, until recently, played a negligible role in physics. One possible explanation of such unpopularity could be that there are multiple non-equivalent definitions of fractional derivatives. Another difficulty is that fractional derivatives have no evident geometrical interpretation because of their non-local character (See L. Debnath [5]). However, during the past several decades fractional calculus has blossomed and grown in pure mathematics as well as in scientific applications because of the fact that, a realistic modelling of a physical phenomenon having dependence not only at the time instant but also the previous time history. In fact, recent advances of fractional calculus are dominated by modern examples of applications in differential and integral equations, plasma physics, image and signal processing, fluid mechanics, viscoelasticity, mathematical biology, electrochemistry and even finance and social sciences.

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations because fractional order models are more accurate than integer order models. However, in general, it is not easy to derive the analytical or exact solutions to most of the fractional differential equations. Therefore, it is vital to develop some reliable and efficient techniques to solve fractional differential equations. In recent decades, several methods have been used to solve fractional differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian decomposition method [10], homotopy perturbation method [11], homotopy analysis method [24], variational iteration method [14], differential transform method [3], finite difference method [23], operational matrix method [8, 21] Haar wavelet method [19, 20], spectral methods [16], B-spline collocation method [12] and many other methods.

In this paper, we start by considering the renowned Lane-Emden Fractional Differential Equation of the type

$$D^{\alpha}y(x)+\frac{c}{x^{\alpha-\beta}}D^{\beta}y(x)+f(x,y)=h(x), 1<\alpha\leq 2, 0<\beta\leq 1 \quad (1.1)$$

together with the initial conditions: $y(0)=c_1 \& y'(0)=c_2$

where $x\in[0,1]$, $c\geq 0, c_1, c_2$ are real constants, $f(x,y)$ is a continuous real valued function and $h(x)\in C[0,1]$[10].

The above fractional Lane-Emden equation utilized to model successfully several real world phenomena in mathematical physics and astrophysics. It demonstrates plenty of phenomena including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents. In addition, the ordinary Lane-Emden equation does not always give a correct description of the dynamics of systems in complex media. Thus, in order to bypass this obstacle and to better describe the dynamical processes in a fractal medium, numerous generalizations of Lane-Emden equation was suggested. Thus, taking into account the memory effects are better described within the fractional derivatives, the fractional Lane-Emden equations are extracting hidden aspects for the complex phenomena they described in various fields of the applied mathematics, mathematical physics, and astrophysics.

In last decades, many researchers sought solutions for fractional Lane-Emden equation by using various methods which includes second kind Chebyshev operational matrix algorithm [7], the modified Legendre-Spectral method [1], the method of Jacobi–Gauss collocation [4], the method of Hermite functions collocation [2], the method of ultraspherical wavelets [22], the method of modified differential transform [13], the method of Legendre multi-wavelets [17] etc. Motivated and inspired by the work of Sahu and Mallick [18],we derive orthonormal Bernoulli's polynomials expansion method for different parameters of the same equation.

The remainder of the paper is organized as follows: In Section 2, we introduce some basic definitions and mathematical preliminaries of fractional calculus. Section 3 depicts the fundamentals of orthonormal Bernoulli's polynomial, its properties and operational matrix of the derivative as a working tool. In Section 4, we derive the function approximation based on the proposed orthonormal Bernoulli's polynomial. Method of solution has been presented in section 5. In section 6 two well known examples of the type fractional order Lane-Emden differential equations are given to demonstrate the efficiency and accuracy of the proposed method. The conclusion is described in the final section.

2. Basic Definitions of Fractional Calculus

In this section, we give some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results.

Definition 2.1. [15]The Riemann-Liouville fractional integration operator of order $\alpha \geq 0$ of a function $f(x)$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, t > 0, \quad (2.1)$$

where $\Gamma(.)$ is the well-known gamma function, and some properties of the operator J^{α} are given as follows:

(i) $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x), \alpha, \beta > 0;$

(ii) $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x), \alpha, \beta > 0;$

(iii) $J^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} x^{\alpha+\beta}, \beta > -1.$

The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by Caputo in his work on the theory of visco-elasticity.

Definition 2.2. The Caputo fractional derivative of D^{α} of a function $f(x)$ is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{m}(\tau)}{(x-\tau)^{\alpha-m+1}}\, d\tau, \quad (2.2)$$

where $m-1 < \alpha \leq m, m \in \mathbb{N}$. Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative.

Similar to integer-order differentiation, the Caputo fractional derivative operator is a linear operator as

$$D^{\alpha}(\gamma f(x) + \delta g(x)) = \gamma D^{\alpha} f(x) + \delta D^{\alpha} g(x),$$

where γ and δ are constants. The Caputo fractional derivative also satisfies the following basic properties:

(i) $D^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, 0 < \alpha < \beta + 1, \beta > -1;$

(ii) $J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0^{+}) \frac{x^k}{k!}, m-1 < \alpha \leq m, m \in \mathbb{N};$

(iii) $D^{\alpha} C = 0, C$ is a constant.

In the present study, the fractional derivatives are considered in the Caputo sense because to obtain a unique solution of a fractional Lane-Emden differential equation, we need to specify additional conditions. For the case of the Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are alike to those of classical differential equations, and are therefore familiar to us.

3. Review of Orthonormal Bernoulli's Polynomial

In this section, we mention some properties of Bernoulli polynomials which will be of fundamental importance in the sequel.

Many researchers have been studied numerical methods based on Bernoulli's polynomials to solve different problems of calculus. The main drawbacks of the Bernoulli's polynomials is that they are not orthogonal. To overcome this problem, we normalize the Bernoulli's polynomials by using Gram-Schmidt orthonormalization process [9] and got an explicit formula of orthonormal Bernoulli's polynomials.

3.1 Definition and properties of orthonormal Bernoulli's polynomials

The Bernoulli's basis polynomials of degree n is denoted by $\mathcal{B}_n(x)$ on $[0,1]$ and are constructed from the following relation

$\sum_{k=0}^{n} \binom{n+1}{k} \mathcal{B}_k(x) = (n+1)x^n, n = 0,1,2,\ldots$ (3.1)

In addition, we can express Bernoulli's polynomial in terms of Bernoulli's numbers as

$\mathcal{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^n,$ (3.2)

where $b_k, k = 0,1,2,\ldots,n$ are Bernoulli's numbers that come from the formula

$$\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{x^i}{i!}, i = 0,1,2,\ldots. (2.4).$$

The first few Bernoulli polynomials are

$$\begin{aligned}\mathcal{B}_0(x) = 1, \mathcal{B}_1(x) = x - \frac{1}{2}, \mathcal{B}_2(x) = x^2 - x + \frac{1}{6}, \mathcal{B}_3(x) \\ = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \mathcal{B}_4(x) \\ = x^4 - 2x^3 + x^2 - \frac{1}{30} (2.5)\end{aligned}$$

Moreover, Bernoulli's polynomials and Bernoulli's numbers satisfy the well-known relations

1. $\mathcal{B}_n(x+1) - \mathcal{B}_n(x) == nx^{n-1}, n \geq 1.$
2. $\mathcal{B}_n(1-x) = (-1)^n \mathcal{B}_n(x), n \geq 1.$
3. $\mathcal{B}'_n(x) = n\mathcal{B}_{n-1}(x), n \geq 1.$
4. $\int_0^1 \mathcal{B}_n(x)dx = 0, n \geq 1.$
5. $\int_0^1 \mathcal{B}_n(x)\mathcal{B}_p(x)dx = (-1)^{n-1} \frac{n!p!}{(n+p)!} b_{n+p}, n, m \geq 1.$
6. $b_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} b_k.$
7. $b_{2n+1} = 0, b_{2n} = \mathcal{B}_{2n}(1).$

Next, by employing the Gram-Schmidt orthonormalization process on the set of Bernoulli's polynomials, we can construct the orthonormal Bernoulli's polynomials, $\psi_n(x)$ which can be expressed explicitly as

$\psi_n(x) = \sqrt{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n-k} x^{n-k}, n = 0,1,2,\ldots$ (3.3)

where the function $\psi_n(x)$ form a basis for $L^2[0,1]$. The orthogonality property satisfies for these polynomials as follows

$\int_0^1 \psi_n(x)\psi_p(x)dx = \delta_{n,p}, n, p = 0,1,2,\ldots$ (3.4)

where $\delta_{n,p}$ is the well- known Kronecker delta function.

4. Function Approximation

A function $f(x) \in L^2[0,1]$ can be expressed by the orthonormal Bernoulli's polynomials as,

$f(x) = \sum_{n=0}^{\infty} a_n \psi_n(x),$ (4.1)

where $a_n = \langle f(x), \psi_n(x) \rangle = \int_0^1 f(x)\psi_n(x)dx$. If the series in the above equation (3.6)is truncated, we then obtain

$f(x) \cong \sum_{n=0}^{N} a_n \psi_n(x) = A^T \Psi(x)$, (4.2)

where A and $\Psi(x)$ are $N+1$ matrices given by $A = [a_0, a_1, \ldots, a_N]^T$,

$\Psi(x) = [\psi_0(x), \psi_1(x), \ldots, \psi_M(x)]^T$.

The suitable collocation points depends on resolution is as follow:

$x_i = \frac{2i-1}{2N}, i = 1,2,\ldots,N.$ (4.3)

Theorem 3.1 If $f(x) \in L^2(\mathbb{R})$ be a continuous function defined on $[0,1]$ and $\| f(x) \| \leq \mathcal{N}_f$, then the orthonormal Bernomial's polynomials expansion of $f(t)$ defined in (3.6) converges uniformly and also

$$\| a_m \| \leq \sqrt{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n-k} \mathcal{N}_f.$$

Proof. The proof is straightforward.

5. Method of Solution

We now ready to construct the solution method of corresponding equation (1.1), as

$D^\alpha y(x) + \frac{c}{x^{\alpha-\beta}} D^\beta y(x) + f(x,y) = h(x), 1 < \alpha \leq 2, 0 < \beta \leq 1$ (5.1)

For this purpose, we first approximate the unknown function $y(x)$, using (4.2) as

$$y(x) = C^T \cdot \Psi(x)$$

$= C^T \cdot A \cdot T_N(x)$, (5.2)

where

$F_N(x) = [1, x, x^2, \ldots, x^N]^T, A = P^{-1}, and\ P = [p_{i,j}],$ (5.3)

where

$p_{i,j} = \sqrt{2j+1} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{2j-k}{j-k} \frac{1}{i+j-k+1}.$ (5.4)

Next,in case of nonlinear term, we assume term

$f(x, y(x)) = q(x)$ (5.5)

and approximate $q(x)$ using (3.2), as

$q(x) = Q^T \cdot \Psi(x) = Q^T \cdot A \cdot F_N(x).$ (5.6)

ISBN-978-93-85822-89-6

Again we expand Caputo-derivative as

$$D^{\alpha}y(x) = D^{\alpha}(C^T \cdot \Psi(x)) = D^{\alpha}(C^T \cdot A.F_N(x))$$

$$= C^T \cdot A \cdot D^{\alpha}(F_N(x)) = C^T \cdot A \cdot N_{\alpha} \cdot x^{-\alpha}F_N(x) \qquad (5.7)$$

where

$$N_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \frac{\Gamma(N)}{\Gamma(N-\alpha)} \end{pmatrix}. \qquad (5.8)$$

In the same way,

$$D^{\beta}y(x) = C^T \cdot A \cdot N_{\beta} \cdot x^{-\beta}F_N(x), \quad (5.9)$$

where

$$N_{\beta} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta)} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\beta)} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \frac{\Gamma(N)}{\Gamma(N-\beta)} \end{pmatrix}. \qquad (5.10)$$

Substituting the matrix relations (5.3),(3.2),(5.7) & (5.9) in (5.1) and simplifying, we obtain the fundamental matrix equation as

$$C^T \cdot A \cdot (N_{\alpha} + cN_{\beta}) \cdot F_N(x) + Q^T \cdot A \cdot x^{-\alpha}F_N(x) = x^{-\alpha}h(x).$$

(5.11)

After applying the collocation points (4.3) in (5.5) & (5.11), we obtain

$$C^T \cdot A \cdot (N_{\alpha} + cN_{\beta}) \cdot F_N(x_i) + Q^T \cdot A \cdot x_i^{-\alpha}F_N(x_i) = x^{-\alpha}h(x_i), \qquad (5.12)$$

and

$$f(x_i, C^T \cdot A \cdot F_N(x_i)) = Q^T \cdot A \cdot F_N(x_i). \qquad (5.13)$$

with initial conditions

$$C^T \cdot A \cdot F_N(0) = c_1, C^T \cdot A \cdot F_N'(0) = c_2. \qquad (5.14)$$

Now, using the equations (5.12),(5.13), & (5.14), we get the system of algebraic equations with $2N$ unknowns $a_0, a_1, \dots, a_n, q_0, q_1, \dots, q_N$. Solving this system of equation numerically to get the value of the matrices $C^T \& Q^T$. By doing all this, we get the approximate solution $y(x)$.

6. Applications

In this section, some numerical problems are given to illustrate the applicability and accuracy of the proposed method. All the numerical computations are carried out using MATLAB.

Example 5.1 Consider the following fractional Lane-Emden Equation:

$$D^{\alpha}y(x) + \frac{1}{x^{\alpha-\beta}}D^{\beta}y(x) + \frac{1}{x^{\alpha-2}}y(x) = h(x), x \in (0,1) \qquad (6.1)$$

where $h(x) = x^{2-\alpha}\left[-6x\left(\frac{\Gamma(4-\beta)+\Gamma(4-\alpha)}{\Gamma(4-\alpha)\Gamma(4-\beta)} + \frac{x^2}{6}\right) + 2\left(\frac{\Gamma(3-\beta)+\Gamma(3-\alpha)}{\Gamma(3-\alpha)\Gamma(3-\beta)} + \frac{x^2}{2}\right)\right]$ for $\alpha = 3/2$ and $\beta = 1$, with subject to initial conditions $y(0) = 1, y'(0) = 0$.

The exact solution of this problem is $y(x) = x^3 - x^2$. The approximate solution by the proposed method for this problem is presented tabularly in Table 5.1. It is clear from the table that the proposed method gives us the accurate values compared with the exact solution.

Table 5.1: Numerical solution of Example 5.1 for $N = 6$ with exact solution.

x	$Exact$	$Approx.$	$Error$
0.1	-0.0091	-0.0091	4.80327×10^{-12}
0.2	-0.0322	-0.0322	4.76248×10^{-12}
0.3	-0.0634	-0.0634	4.87696×10^{-12}
0.4	-0.0967	-0.0967	4.98397×10^{-12}
0.5	-0.1258	-0.1258	4.23442×10^{-12}
0.6	-0.1449	-0.1449	4.43918×10^{-12}
0.7	-0.1474	-0.1474	4.75914×10^{-12}
0.8	-0.1283	-0.1283	4.34519×10^{-12}
0.9	-0.0813	-0.0813	4.76821×10^{-12}

Example 5.2 Consider the following Lane-Emden fractional differential Equation:

$$D^{\alpha}y(x) + \frac{1}{x^{\alpha-\beta}}D^{\beta}y(x) + \frac{1}{1-x}y(x) = h(x), x \in (0,1) \qquad (6.2)$$

where $h(x) = \frac{x^3}{1-x}\cos(x) - 5x\sin(x)4\cos(x)$ for $\alpha = 1.9$ and $\beta = 0.9$, with subject to initial conditions $y(0) = 0, y'(0) = 0$.

The exact solution of this problem is $y(x) = x^2\cos(x)$. The approximate solution by the proposed method for this problem is presented tabularly in Table 5.2. It is clear from the table that the proposed method gives us the accurate values compared with the exact solution.

Table 5.2: Numerical solution of Example 5.2 for $N = 6$with exact solution.

x	$Exact$	$Approx.$	$Error$
0.1	0.0099	0.0099	4.84324×10^{-14}
0.2	0.3999	0.3999	4.8624×10^{-14}
0.3	0.0899	0.0899	4.8769×10^{-13}
0.4	0.1599	0.1599	4.1239×10^{-14}
0.5	0.2499	0.2499	4.6744×10^{-14}
0.6	0.3599	0.3599	4.9891×10^{-14}
0.7	0.4898	0.4898	4.9891×10^{-12}
0.8	0.6399	0.6399	4.0050×10^{-13}
0.9	0.8099	0.8099	4.8082×10^{-14}

7. Conclusion

We construct a Bernoulli's polynomial expansion for solving fractional order Lane-Emden differential equations. Also, we have given a general procedure of forming the operational matrix N_α and N_β which plays an important role in this technique. The main advantage of the present techniques is that it transforms the problem into algebraic system of equation so that the computation is quite easy and simple. Two well known examples have been considered to check the reliability and effectiveness of the proposed method. Moreover, we compared the approximate result with the exact solution reported recently in the literature.

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ISBN-978-93-85822-89-6
Volume II